Machine Learning (SS2015)

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Exercise Sheet 2
Exercise 2.1: Perceptrons

Given is a perceptron with weight vector \((w_0, w_1, w_2)^T = (2, 1, 1)^T\).

(a) Plot the partition of \(\mathbb{R}^2\) that is realized by this perceptron in a diagram and mark the area where the perceptron outputs 1.

\[
y = f_{\text{step}} (w_0 + \langle \vec{w}, \vec{x} \rangle)
\]
(b) Which of the perceptrons with the following weight vectors have the same hyperplane and which represent exactly the same classification as the perceptron given above?

(I) \((w_0, w_1, w_2)^T = (1, 0.5, 0.5)^T\)

(II) \((w_0, w_1, w_2)^T = (200, 100, 100)^T\)

(III) \((w_0, w_1, w_2)^T = (\sqrt{2}, \sqrt{1}, \sqrt{1})^T\)

(IV) \((w_0, w_1, w_2)^T = (-2, -1, -1)^T\)

- (I) and (II) have different length than \((w_0, w_1, w_2)^T\), but point into the same direction. They realize the same hyperplane and represent the same classification.
- (III) points into a different direction and therefore realizes a different hyperplane.
- (IV) points into the opposite direction. The hyperplane is the same, but the classification is not.
Exercise 2.2: Perceptron Learning

(a) Apply the perceptron learning algorithm for the following pattern set until convergence. Start with weight vector \((w_0, w_1, w_2, w_3)^T = (1, 0, 0, 0)^T\). Apply the patterns in the given order cyclically. For each step of perceptron learning write down the applied pattern, the classification result and the update of the weight vector.

\[
\begin{align*}
(4, 3, 6)^T & \in \mathcal{N} \\
(2, -2, 3)^T & \in \mathcal{P} \\
(1, 0, -3)^T & \in \mathcal{P} \\
(4, 2, 3)^T & \in \mathcal{N}
\end{align*}
\]
<table>
<thead>
<tr>
<th>pattern</th>
<th>output</th>
<th>classification</th>
<th>update</th>
<th>new weight vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 4, 3, 6)^T \in \mathcal{N}$</td>
<td>$f_{step}(1)$</td>
<td>false positive</td>
<td>$-(1, 4, 3, 6)^T$</td>
<td>$(0, -4, -3, -6)^T$</td>
</tr>
<tr>
<td>$(1, 2, -2, 3)^T \in \mathcal{P}$</td>
<td>$f_{step}(-20)$</td>
<td>false negative</td>
<td>$+(1, 2, -2, 3)^T$</td>
<td>$(1, -2, -5, -3)^T$</td>
</tr>
<tr>
<td>$(1, 4, 2, 3)^T \in \mathcal{N}$</td>
<td>$f_{step}(-26)$</td>
<td>true negative</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 4, 3, 6)^T \in \mathcal{N}$</td>
<td>$f_{step}(-40)$</td>
<td>true negative</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 2, -2, 3)^T \in \mathcal{P}$</td>
<td>$f_{step}(-2)$</td>
<td>false negative</td>
<td>$+(1, 2, -2, 3)^T$</td>
<td>$(2, 0, -7, 0)^T$</td>
</tr>
<tr>
<td>$(1, 1, 0, -3)^T \in \mathcal{P}$</td>
<td>$f_{step}(2)$</td>
<td>true positive</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 4, 2, 3)^T \in \mathcal{N}$</td>
<td>$f_{step}(-12)$</td>
<td>true negative</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 4, 3, 6)^T \in \mathcal{N}$</td>
<td>$f_{step}(-19)$</td>
<td>true negative</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 2, -2, 3)^T \in \mathcal{P}$</td>
<td>$f_{step}(16)$</td>
<td>true positive</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
</tbody>
</table>

finished, weight vector $(2, 0, -7, 0)^T$ classifies all patterns correctly
(b) Show that the problem given by the following pattern set cannot be solved with a single perceptron. For this purpose, apply the perceptron learning algorithm for the given patterns. Start with weight vector \((w_0, w_1, w_2)^T = (1, 0, 0)^T\).

\[
\begin{align*}
(1, 1)^T & \in \mathcal{P} \\
(1, 0)^T & \in \mathcal{N} \\
(0, 0)^T & \in \mathcal{P} \\
(0, 1)^T & \in \mathcal{N}
\end{align*}
\]
<table>
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<th>new weight vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1)^T$ ∈ $\mathcal{P}$</td>
<td>$f_{step}(1)$</td>
<td>true positive</td>
<td>unchanged</td>
<td>$(1, 0, 0)^T$</td>
</tr>
<tr>
<td>$(1, 1, 0)^T$ ∈ $\mathcal{N}$</td>
<td>$f_{step}(1)$</td>
<td>false positive</td>
<td>$-(1, 1, 0)^T$</td>
<td>$(0, -1, 0)^T$</td>
</tr>
<tr>
<td>$(1, 0, 0)^T$ ∈ $\mathcal{P}$</td>
<td>$f_{step}(0)$</td>
<td>true positive</td>
<td>unchanged</td>
<td>unchanged</td>
</tr>
<tr>
<td>$(1, 0, 1)^T$ ∈ $\mathcal{N}$</td>
<td>$f_{step}(0)$</td>
<td>false positive</td>
<td>$-(1, 0, 1)^T$</td>
<td>$(-1, -1, -1)^T$</td>
</tr>
<tr>
<td>$(1, 1, 1)^T$ ∈ $\mathcal{P}$</td>
<td>$f_{step}(-3)$</td>
<td>false negative</td>
<td>$+(1, 1, 1)^T$</td>
<td>$(0, 0, 0)^T$</td>
</tr>
<tr>
<td>$(1, 1, 0)^T$ ∈ $\mathcal{N}$</td>
<td>$f_{step}(0)$</td>
<td>false positive</td>
<td>$-(1, 1, 0)^T$</td>
<td>$(-1, -1, 0)^T$</td>
</tr>
<tr>
<td>$(1, 0, 0)^T$ ∈ $\mathcal{P}$</td>
<td>$f_{step}(-1)$</td>
<td>false negative</td>
<td>$+(1, 0, 0)^T$</td>
<td>$(0, -1, 0)^T$</td>
</tr>
</tbody>
</table>

finished, weight vector $(0, -1, 0)^T$ occurs twice
the problem is not solvable (cycle theorem)
Exercise 2.3: Perceptron Networks

Develop a perceptron network with two input variables $x_1$ and $x_2$ and at most three perceptrons that exactly classifies the marked area (and the boundary) plotted in Figure 1 as positive. Illustrate the topology (structure) of the network and the weights of its neurons.

Figure 1: Input space of the perceptron network. The gray area is classified as positive.
Exercise 2.4: Nonlinear Feature Spaces

The following training patterns are given:

\[ (-3) \in \mathcal{N}, (-2) \in \mathcal{N}, (-1) \in \mathcal{P}, (0) \in \mathcal{P}, (1) \in \mathcal{P}, (2) \in \mathcal{N}, (3) \in \mathcal{N} \]

This pattern set is not linearly separable in \( \mathbb{R} \). We will use nonlinear features in order to classify the patterns using a single perceptron.

(a) Plot the patterns in input space and mark each positive pattern with output 1 and each negative pattern with output 0.

\[
\begin{array}{cccccccc}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\bullet & & & & & \bullet & & \bullet & \\
\text{blue: positive, red: negative}
\end{array}
\]
(b) Use the function \((h_1, h_2)^T = g(x) = (x, x^2)^T\) to build a feature space for the patterns. Plot the patterns in feature space and mark each positive pattern with output 1 and each negative pattern with output 0. Show that the patterns are linearly separable in feature space (give a perceptron that classifies correctly) or derive a contradiction from the data and the model.

\[
\begin{align*}
\text{blue: positive, red: negative} \\
w &= (2, 0, -1)^T \text{ classifies all patterns correctly}
\end{align*}
\]
(c) Use the function \((h_1, h_2) = g(x) = (x, x^3)^T\) to build a feature space for the patterns. Plot the patterns in feature space. Are the patterns linearly separable in feature space?

blue: positive, red: negative

use cycle theorem to prove that the data is not linearly separable