Exercise 3.1: Perceptrons

Given is a perceptron with weight vector \((w_0, w_1, w_2)^T = (2, 1, 1)^T\).

(a) Plot the partition of \(\mathbb{R}^2\) that is realized by this perceptron in a diagram and mark the area where the perceptron outputs 1.

(b) Which of the perceptrons with the following weight vectors have the same hyperplane and which represent exactly the same classification as the perceptron given above?

\begin{align*}
(I) \quad (w_0, w_1, w_2)^T &= (1, 0.5, 0.5)^T \\
(II) \quad (w_0, w_1, w_2)^T &= (200, 100, 100)^T \\
(III) \quad (w_0, w_1, w_2)^T &= (\sqrt{2}, \sqrt{1}, \sqrt{1})^T \\
(IV) \quad (w_0, w_1, w_2)^T &= (-2, -1, -1)^T
\end{align*}

Exercise 3.2: Perceptron Learning

(a) Apply the perceptron learning algorithm for the following pattern set until convergence. Start with weight vector \((w_0, w_1, w_2, w_3)^T = (1, 0, 0, 0)^T\). Apply the patterns in the given order cyclically. For each step of perceptron learning write down the applied pattern, the classification result and the update of the weight vector.

\begin{align*}
(4, 3, 6)^T \quad &\in N \\
(2, -2, 3)^T \quad &\in P \\
(1, 0, -3)^T \quad &\in P \\
(4, 2, 3)^T \quad &\in N
\end{align*}

(b) Show that the problem given by the following pattern set cannot be solved with a single perceptron. For this purpose, apply the perceptron learning algorithm for the given patterns. Start with weight vector \((w_0, w_1, w_2)^T = (1, 0, 0)^T\).

\begin{align*}
(1, 1)^T \quad &\in P \\
(1, 0)^T \quad &\in N \\
(0, 0)^T \quad &\in P \\
(0, 1)^T \quad &\in N
\end{align*}
Exercise 3.3: Perceptron Networks

Develop a perceptron network with two input variables $x_1$ and $x_2$ and at most three perceptrons that exactly classifies the marked area (and the boundary) plotted in Figure 1 as positive. Illustrate the topology (structure) of the network and the weights of its neurons.

Figure 1: Input space of the perceptron network. The gray area is classified as positive.

Exercise 3.4: Nonlinear Feature Spaces

The following training patterns are given:

$\begin{align*}
\begin{array}{c}
( -3 ) \in \mathcal{N}, ( -2 ) \in \mathcal{N}, ( -1 ) \in \mathcal{P}, ( 0 ) \in \mathcal{P}, ( 1 ) \in \mathcal{P}, ( 2 ) \in \mathcal{N}, ( 3 ) \in \mathcal{N}
\end{array}
\end{align*}$

This pattern set is not linearly separable in $\mathbb{R}$. We will use nonlinear features in order to classify the patterns using a single perceptron.

(a) Plot the patterns in input space and mark each positive pattern with output 1 and each negative pattern with output 0.

(b) Use the function $(h_1, h_2)^T = g(x) = (x, x^2)^T$ to build a feature space for the patterns. Plot the patterns in feature space and mark each positive pattern with output 1 and each negative pattern with output 0. Show that the patterns are linearly separable in feature space (give a perceptron that classifies correctly) or derive a contradiction from the data and the model.

(c) Use the function $h = g(x) = x^3$ to build a feature space for the patterns. Plot the patterns in feature space. Are the patterns linearly separable in feature space?